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ON A TELEPHONE TRAFFIC SYSTEM WITH SEVERAL KINDS OF SERVICE DISTRIBUTIONS

by

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§1 Introduction

Let us suppose that at a telephone exchange calls are arriving at the instants $\tau_1, \tau_2, \tau_3, \ldots, \tau_n, \ldots,$ where $0 < \tau_1 < \tau_2 < \ldots < \tau_n < \ldots < \infty$. As usual, in queueing theory, we assume that the inter-arrival times $\theta_n = \tau_n - \tau_{n-1}$ $(n=1,2,\ldots,\tau_0=0)$ are identically distributed, independent, positive random variables with distribution function

$$1) \qquad P\{\theta_n \leq x\} = F(x), \quad (n = 1,2,\ldots) .$$

Let

$$2) \qquad \alpha = \int_0^\infty x dF(x),$$

and

$$\varphi(s) = \int_0^\infty e^{-sx} dF(x) .$$

The input is said to be a recurrent process. We shall assume that there are infinitely many lines available and that, therefore, no call is ever lost. Ordinarily, it is assumed that the holding times (the durations of the connections) are identically distributed, independent, random variables with an exponential distribution. The American folklore, however, assumes that women talk more than men. In this paper, therefore, we shall consider the possibility of calls requiring one of several (s) different exponential holding times.

For simplicity, let us consider the case where $s = 2$. Thus, when a call arrives, it will be either of the first kind with probability p, or of the second kind with probability q. The holding

times of calls of the first kind have the distribution function

4) $$H_1(x) = 1-e^{-\lambda x} .$$

The holding times of calls of the second kind have the distribution function

5) $$H_2(x) = 1-e^{-\mu x} .$$

We let $(\xi(t), \eta(t))$ be a vector random variable denoting the number of calls of the first and second kinds respectively present in the system at time t. We define (ξ_n, η_n) to be $(\xi(\tau_n-0), \eta(\tau_n-0))$, i.e., (ξ_n, η_n) is the state of the system immediately before the n-th arrival. Thus, the system is in state E_{jk} at time t if $(\xi(t), \eta(t)) = (j, k)$.

We shall obtain the binomial moments of the limiting distribution of the imbedded Markov chain (ξ_n, η_n). In order to accomplish this, we first give a bivariate extension of Jordan's inversion formula.

Later, we consider the general case where we allow an arriving call to have one of the s types of holding time distributions with probability p_i ($\sum_{i=1}^{s} p_i = 1$). Each of the holding time distributions is assumed to be exponential, i.e.,

6) $$H_i(x) = 1-e^{-\mu_i x} \quad (i = 1,2,\ldots,s).$$

In section 5, we show that the dual problem of a Poisson input with the possibility of several different holding time distributions is easily reduced to the $M/G/\infty$ case discussed by Takács [1] .

It is well known that the probability theory of Type II particle counters is intimately connected with that of the queue with infinitely many servers. We use our results to derive the mean time between consecutive registrations in a Type II counter when the particles arrive according to a recurrent process and the durations of the impulses produced are distributed as a weighted sum of exponential random variables.

§2. Extensions of Jordan's Inversion Formula.

If $\{P_k\}$ $(k = 0,1,2, \ldots)$ is a discrete probability distribution, then the r-th binomial moment is defined by

$$B_r = \sum_{k=r}^{\infty} \binom{k}{r} P_k \quad . \tag{7}$$

If the generating function $U(z) = \sum_{k=0}^{\infty} P_k z^k$ is analytic in a circle of radius $1+\epsilon$, where ϵ can be an arbitrarily small positive number, then

$$B_r = \frac{1}{r!} \left. \frac{d^r U(z)}{dz^r} \right|_{z=1} \tag{8}$$

and the binomial moments uniquely determine the distribution $\{P_k\}$. Jordan's inversion formula expresses the $P_k (k = 0,1,2, \ldots)$ in terms of the $B_r (r = 0,1,2, \ldots)$.

Theorem 1 (Jordan).

9) $$P_k = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} B_r \quad .$$

If

$$\{P_{k_1 k_2}\} \quad (k_1 = 0,1,\ldots, k_2 = 0,1,\ldots)$$

is a discrete bivariate distribution, we define the $r_1 r_2$-th binomial moment by

10) $$B_{r_1 r_1} = \sum_{k_1=r_1}^{\infty} \sum_{k_2=r_2}^{\infty} \binom{k_1}{r_1}\binom{k_2}{r_2} P_{k_1 k_2} \; .$$

If $$U(z,w) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} P_{k_1 k_2} z^{k_1} w^{k_2}$$

is analytic in the region $|z| < 1+\epsilon$, $|w| < 1+\epsilon$, then

11) $$B_{r_1 r_2} = \frac{1}{r_1! r_2!} \left. \frac{\partial^{r_1+r_2} U(z,w)}{\partial z^{r_1} \partial z^{r_2}} \right|_{(1,1)} \quad .$$

The fundamental inversion formula now becomes

12) $$P_{k_1 k_2} = \sum_{r_1=k_1}^{\infty} \sum_{r_2=k_2}^{\infty} (-1)^{r_1-k_1} (-1)^{r_2-k_2} \binom{r_1}{k_1}\binom{r_2}{k_2} B_{r_1 r_2} \; .$$

Before proceeding with the proof, I shall state the general multivariate result which is proved similarly.

Let

$$P_{k_1 k_2 \ldots k_n} \quad (k_1 = 0,1,2,\ldots,\ldots, k_n = 0,1,2,\ldots)$$

be a discrete n-variate distribution. Defining the $(r_1 r_2 \ldots r_n)$-th binomial moment by

13) $$B_{r_1 r_2 \ldots r_n} = \sum_{k_1=r_1}^{\infty} \sum_{k_2=r_2}^{\infty} \cdots \sum_{k_n=r_n}^{\infty} \binom{k_1}{r_1}\binom{k_2}{r_2} \cdots \binom{k_n}{r_n} P_{k_1 k_2 \ldots k_n} \; ,$$

then the proper inversion formula is seen to be

(14)

$$P_{k_1k_2\dots k_n}=\sum_{r_1=k_1}^{\infty}\sum_{r_2=k_2}^{\infty}\cdots\sum_{k_n=r_n}^{\infty}(-1)^{(r_1-k_1)+(r_2-k_2)+\dots+(r_n-k_n)}\prod_{i=1}^{n}\binom{r_i}{k_i}B_{r_1r_2\dots r_n}\,.$$

Proof of the Bivariate Case.

In order to verify the assertion that

$$\text{15)}\qquad P_{k_1k_2}=\sum_{r_1=k_1}^{\infty}\sum_{r_2=k_2}^{\infty}(-1)^{r_1-k_1}(-1)^{r_2-k_2}\binom{r_1}{k_1}\binom{r_2}{k_2}B_{r_1r_2}\quad,$$

we substitute the value of $B_{r_1r_2}$ in the right side of (15), obtaining

$$\text{16)}\ \sum_{r_1=k_1}^{\infty}\sum_{r_2=r_2}^{\infty}(-1)^{r_1-k_1}(-1)^{r_2-k_2}\binom{r_1}{k_1}\binom{r_2}{k_2}\left\{\sum_{m=r_1}^{\infty}\sum_{n=r_2}^{\infty}\binom{m}{r_1}\binom{n}{r_2}P_{mn}\right\}$$

$$=\sum_{r_1=k_1}^{\infty}\sum_{r_2=k_2}^{\infty}\sum_{m=r_1}^{\infty}\sum_{n=r_2}^{\infty}(-1)^{r_1-k_1}(-1)^{r_2-k_2}\binom{r_1}{k_1}\binom{r_2}{k_2}\binom{m}{r_1}\binom{n}{r_2}P_{mn}\,.$$

Using the fact that

$$\binom{r}{k_1}\binom{m}{r_1}=\binom{m}{k_1}\binom{m-k_1}{r_1-k_1}$$

and

$$\binom{r_2}{k_2}\binom{n}{r_2}=\binom{n}{k_2}\binom{n-k_2}{r_2-k_2}\quad,$$

interchanging the sums

$$\sum_{m=k_1}^{\infty}\sum_{r_1-k_1=0}^{\infty}\quad\text{and}\quad\sum_{n=k_2}^{\infty}\sum_{r_2-k_2=0}^{n-k_2}$$

yields

$$17) \quad \sum_{m=k_1}^{\infty} \sum_{n=k_2}^{\infty} \binom{m}{k_1}\binom{n}{k_2} p_{mn} \left\{ \sum_{r_1-k_1=0}^{m-k_1} (-1)^{r_1-k_1} \binom{m-k_1}{r_1-k_1} \cdot \sum_{r_2-k_2=0}^{n-k_2} (-1)^{r_2-k_2} \binom{n-k_2}{r_2-k_2} \right\} .$$

Recall, however, that $(-1)^{n-k} = \begin{cases} 0 & \text{if } n > k \\ 1 & \text{if } n = k \end{cases}$.

Thus, the only non-zero term in our sum occurs when $n=k_2$ and $m=k_1$. Then our sum reduces to

$$18) \qquad \sum_{(m=k_1, n=k_2)} p_{mn} = p_{k_1 k_2} \quad .$$

§3. The Ergodic Behavior of the Imbedded Markov Chain (ξ_n, η_n) .

If we look at our process at the times just before a call arrives, then the random variables $(\xi_n, \eta_n) = (\xi(\tau_n - 0), \eta(\tau_n - 0))$ form a Markov Chain because we have assumed exponential holding times. The transition probabilities are given by

$$19) \quad P_{jk\ell m} = P\left[(\xi_{n+1}, \eta_{n+1}) = (\ell, m) \mid (\xi_n, \eta_n) = (j, k)\right] ,$$

$$= p \int_0^{\infty} \binom{j+1}{\ell} e^{-\lambda \ell x} (1-e^{-\lambda x})^{j+1-\ell} \binom{k}{m} e^{-\mu x m} (1-e^{-\mu x})^{k-m} dF(x)$$

$$+ q \int_0^{\infty} \binom{j}{\ell} e^{-\lambda \ell x} (1-e^{-\lambda x})^{j-\ell} \binom{k+1}{m} e^{-\mu m x} (1-e^{-\mu x})^{m+1-k} dF(x) .$$

Starting from the initial distribution $\{P_k^{(1)}\}$, the distributions $\{P_k^{(n)}\}$ can be determined successively from the Chapman-Kolmogoroff equations

$$20) \qquad P_{\ell m}^{(n+1)} = \sum_{j,k} P_{jk\ell m} P_{jk}^{(n)} \quad .$$

It is known, however, that it is much more convenient to work with the binomial moments of the distribution (ξ_n, η_n), i.e.,

21) $$B_{r_1 r_2}^{(n)} = E\left\{\binom{\xi_n}{r_1}\binom{\eta_n}{r_2}\right\} = \sum_{j=r_1}^{\infty} \sum_{k=r_2}^{\infty} \binom{j}{r_1}\binom{k}{r_2} p_{jk}^{(n)} .$$

<u>Lemma.</u> <u>Let</u> X <u>and</u> Y <u>be two independent binomial distributions with parameters</u> (p_1, n_2) (p_2, n_2) <u>respectively.</u> <u>Let</u> $P_{k_1 k_2} = P\{X=k_1, Y=k_2\}$ <u>be their joint distribution.</u> <u>Then</u>

22) $$B_{r_1 r_2} = \sum_{k_1=r_1}^{n_1} \sum_{k_2=r_2}^{n_2} \binom{k_1}{r_1}\binom{k_2}{r_2} P_{k_1 k_2} = \binom{n_1}{r_1}\binom{n_2}{r_2} p_1^{r_1} p_2^{r_2} .$$

The proof is purely computational.

<u>Theorem 2.</u> <u>We have</u> $B_{00}^{(n)} = 1$ $(n = 1, 2, \ldots)$ <u>and</u>

23) $$B_{r_1 r_2}^{(n+1)} = \varphi(r_1\lambda + r_2\mu) \left\{B_{r_1 r_2}^{(n)} + pB_{r_1-1, r_2}^{(n)} + qB_{r_1, r_2-1}^{(n)}\right\} .$$

<u>Proof.</u> By our lemma,

$$E\left\{\binom{\xi_{n+1}}{r_1}\binom{\eta_{n+1}}{r_2} \mid (\xi_n, \eta_n) = (j,k),\ \theta_n = x\right\} =$$

24) $$p\binom{j+1}{r_1} e^{-r_1\lambda x} \binom{k}{r_2} e^{-r_2\mu x} + q\binom{j}{r_1} e^{-r_1\lambda x} \binom{k+1}{r_2} e^{-r_2\mu x} ,$$

because under our conditions the call is of the first kind, with probability p, and thus

ξ_{n+1} is binomial with parameters $j+1$ and $e^{-\lambda x}$

and

η_{n+1} is binomial with parameters k and $e^{-\mu x}$.

Likewise, the call is of the second kind, with probability q, and thus

ξ_{n+1} is binomial with parameters j and $e^{-\lambda x}$

and

η_{n+1} is binomial with parameters $k+1$ and $e^{-\mu x}$.

Therefore, we have proved (24). Removing the conditioning on θ_n yields

$$25)\quad E\left\{\binom{\xi_{n+1}}{r_1}\binom{\eta_{n+1}}{r_2}\Big|(\xi_n,\eta_n)=(j,k)\right\}= \varphi(r_1\lambda+r_2\mu)\left[p\binom{j+1}{r_1}\binom{k}{r_2}+q\binom{j}{r_1}\binom{k+1}{r_2}\right] .$$

Multiplying both sides of (25) by $P_{jk}^{(n)}$ and summing over all relevant (j,k), we obtain

$$26)\quad B_{r_1r_2}^{(n+1)} = \varphi(r_1\lambda+r_2\mu)\left[B_{r_1r_2}^{(n)} + pB_{r_1-1,r_2}^{(n)} + qB_{r_1,r_2-1}^{(n)}\right] .$$

It is clear from (26) that if the limiting distribution $P_{jk} = \lim_{n\to\infty} P\left[(\xi_n,\eta_n) = (j,k)\right]$ exists (which it does in our case), then the binomial moments of $\{P_{jk}\}$ satisfy the equation

$$27)\quad B_{r_1r_2} = \frac{\varphi(r_1\lambda+r_2\mu)}{1-\varphi(r_1\lambda+r_2\mu)}\left[pB_{r_1-1,r_2} + qB_{r_1,r_2-1}\right],$$

with $B_{00} = 1$

Before presenting the solution to the difference equations above we introduce further notation. Let $D(r_1,r_2)$ denote the set of all decreasing paths from (r_1,r_2) to $(0,0)$. A decreasing path is one that always goes down or to the left. Let

$$C(m,n) = \frac{\varphi(m\lambda+n\mu)}{1-\varphi(m\lambda+n\mu)}$$

if either m or n is different from zero; and let $C(0,0) = 1$. For any path ρ in $D(r_1,r_2)$, form

$$C(\rho) = \prod C(m_1,m_2),$$

where the product is taken over all points (m_1,m_2) of the path.

Theorem 3. The solution to the system of equations (27) is

$$B_{r_1r_2} = p^{r_1}q^{r_2}\left\{\sum_{\rho\in D(r_1,r_2)} C(\rho)\right\} . \tag{28}$$

The proof can be accomplished by double induction but a glance at the equation shows that the $B_{r_1r_2}$ given by (28) fit the equation and reduce to the results of Takács if $p = 1$. In his case there is only one decreasing path as he dealt with a 1-dimensional problem and $B_r = \prod_{j=1}^{r} C_j$. If we impose the initial conditions

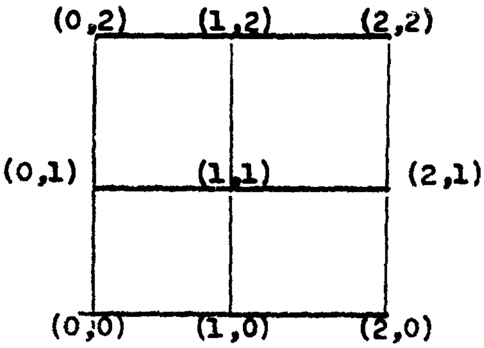
$$B_{r_10} = \prod_{j=1}^{r_1} \frac{\varphi(j\lambda)}{1-\varphi(j\lambda)}$$

and

$$B_{0r_2} = \prod_{j=1}^{r_2} \frac{\varphi(j\mu)}{1-\varphi(j\mu)},$$

we see that (28) gives the unique solution to our equation.

As an example we compute B_{22}.



There are 6 possible paths from (2,2) to (0,0). Namely,

(2,2)-(2,1)-(2,0)-(1,0)-(0,0)
(2,2)-(1,2)-(0,2)-(0,1)-(0,0)
(2,2)-(2,1)-(1,1)-(1,0)-(0,0)
(2,2)-(2,1)-(1,1)-(0,1)-(0,0)
(2,2)-(1,2)-(1,1)-(1,0)-(0,0)
(2,2)-(1,2)-(1,1)-(0,1)-(0,0)

Consequently,

$$B_{22} = C_{22}C_{12}C_{02}C_{01} + C_{22}C_{12}C_{11}C_{01} + C_{22}C_{12}C_{11}C_{10}$$
$$+ C_{22}C_{21}C_{20}C_{10} + C_{22}C_{21}C_{11}C_{01} + C_{22}C_{21}C_{11}C_{10} .$$

§4. The General S-Dimensional Situation.

We now permit an incoming call to have one of s exponential holding time distributions. We deal with the following model: Calls arrive at a telephone exchange at times $\tau_1 < \tau_2 < \ldots < \tau_n < \ldots$. The inter-arrival times $\theta_n = \tau_n - \tau_{n-1}$ $(n = 1,2, \ldots)$ are mutually independent, identically distributed, random variables. Thus,

29) $$P\left[\theta_n \leq x\right] = F(x) \quad (n = 1,2, \ldots) ,$$

30) $$\alpha = \int_0^\infty x dF(x) < \infty ,$$

and

31) $$\varphi(s) = \int_0^\infty e^{sx} dF(x) .$$

Each call has probability p_i of having the i-th kind of holding time distribution

32) $$H_i(x) = 1-e^{-\mu_i x} \quad (i = 1, \ldots, s) .$$

Of course, $\sum_{i=1}^{s} p_i = 1$. As usual, we assume that holding time distributions chosen are mutually independent random variables and independent of the sequence of times of arrival $\{\tau_n\}$.

We denote the state of our system at time t by an s-vector $[\xi^1(t), \xi^2(t), \ldots, \xi^s(t)]$ where $\xi^i(t)$ is the number of calls of the i-th kind present in the system at time t . We shall determine the binomial moments of the imbedded Markov chain of this process.

Let $\bar{\xi}_n = (\xi_n^1, \ldots, \xi_n^s) = [\xi_n^1(\tau_n - 0), \ldots, \xi_n^s(\tau_n - 0)]$.
The transition probabilities of the s-dimensional chain are given by

33) $$p_{(j_1,\ldots,j_s)(k_1,\ldots,k_s)} = P[\bar{\xi}_{n+1} = (k_1,\ldots,k_s) | \bar{\xi}_n = (j_1,\ldots,j_s)]$$

$$= \sum_{i=1}^{s} p_i \int_0^\infty \binom{j_i+1}{k_i} e^{-\mu_i k_i}(1-e^{-\mu_i x})^{j_i+1-k_i} \prod_{v\neq i} \binom{j_v}{k_v} e^{-\mu_v x k_v}\left(1-e^{-\mu_v x}\right)^{j_v-k_v} dF(x)$$

As before, it is easier to work with binomial moments

34) $$B^{(n)}_{r_1 r_2 \ldots r_s} = E\left\{\binom{\xi_n^1}{r_1}\binom{\xi_n^2}{r_2}\cdots\binom{\xi_n^s}{r_s}\right\}$$

$$= \sum_{j_1=r_1}^{\infty} \sum_{j_2=r_2}^{\infty} \cdots \sum_{j_s=r_s}^{\infty} \prod_{i=1}^{s}\binom{j_i}{r_i} P^{(n)}_{j_1\ldots j_s} .$$

Proceeding as before, we arrive at the following equation satisfied by binomial moments of the stationary distribution

$$P_{j_1\ldots j_s} = \lim_{n\to\infty} P^{(n)}_{j_1\ldots j_s} :$$

$$35)\ B_{r_1 r_2\ldots r_s} = \frac{\varphi(\sum_{i=1}^{s} r_i\mu_i)}{1-\varphi(\sum_{i=1}^{s} r_i\mu_i)}\Big[p_1 B_{r_1-1,r_2,\ldots,r_s} + p_2 B_{r_1,r_2-1,r_2,\ldots r_s} + \ldots + p_s B_{r_1 r_2\ldots r_{s-1},r_s-1}\Big]$$

with

$$B_{00\ldots 0} = 1 .$$

Let $D(r_1, \ldots ,r_s)$ denote the set of all decreasing paths from $(r_1,\ldots,r_s)$ to $(0, \ldots , 0)$, and let

$$C(m_1,\ldots,m_s) = \frac{\varphi(\sum_{i=1}^{s} m_i\mu_i)}{1-\varphi(\sum_{i=1}^{s} m_i\mu_i)} .$$

For any path ρ in $D(r_1,\ldots,r_s)$ form

$$C(\rho) = \prod C(m_1,\ldots,m_s),$$

where the product is taken over all points $(m_1,\ldots,m_s)$ of the path ρ. We now can give an explicit formula for the moments of the stationary distribution $\{P_{j_1\ldots j_s}\}$.

<u>Theorem 4. The solution to the system of equations</u> (35) <u>is</u>

$$(36)\qquad B_{r_1,\ldots,r_s} = \prod_{i=1}^{s} p_i^{r_i}\Big\{\sum_{\rho\in D(r_1,\ldots,r_s)} C(\rho)\Big\} .$$

§5. Remarks on the Poisson Input Case.

Suppose the calls arrive according to a Poisson process. Each call will have probability p_i of having one of the s holding time distributions

$$H_i(x) \; \sum_{i=1}^{s} p_i = 1). \qquad (i=1, \ldots ,s),$$

The only requirement on the holding time distributions is that they have finite mean, i.e.,

$$\alpha_i = \int_0^{\infty} x dH_i(x) = \int_0^{\infty} [1-H_i(x)]dx < \infty \qquad \text{for all } i .$$

This problem is trivial because we can consider the input to be composed of s different Poisson processes each with parameter λp_i. Thus, we can regard our system as s different $M/G/\infty$ systems. If we let $\xi = (\xi^1(t), \ldots ,\xi^s(t))$ denote the state of our system, then by Theorem 11 of Reference [1] (also Theorem 1 page 160 of [2]), we have

$$37) \qquad P[\xi(t) = (k_1, \ldots ,k_s)] = e^{-\sum_{i=1}^{s} g_i} \prod_{i=1}^{s} \frac{g_i^{k_i}}{k_i!} ,$$

where

$$g_i = \lambda p_i \int_0^t [1-H_i(x)]dx \qquad (i = 1, \ldots ,s).$$

§6. A Related Particle Counting Problem.

In this section we analyse our problem from the point of view of particle counting. We assume that particles arrive at a Type II counter at times $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ $(n=1,2,\ldots; \tau_0=0)$, where the inter-arrival times $\theta_n = \tau_{n+1} - \tau_n$ are identically distributed, positive, random variables with distribution function $F(x)$. As before, we let

$$\alpha = \int_0^\infty x dF(x)$$

and

$$\varphi(s) = \int_0^\infty e^{-sx} dF(x).$$

At the instant of its arrival a particle produces one of s kinds of impulses. Each particle has probability p_i of producing the i-th kind of impulse the duration of which has the distribution function

$$H_i(x) = 1-e^{-\mu_i x} \quad (i = 1,\ldots,s). \tag{38}$$

Of course, $\sum_{i=1}^{s} p_i = 1$. Furthermore, we assume that the durations of the impulse times which are chosen are mutually independent, random variables that are also independent of the sequence of times of arrival $\{\tau_n\}$.

Although every arriving particle produces an impulse in the counter, only those particles that arrive when the counter is free are registered. The times of arrival of the registered particles

$\{\tau_n'\}$ form a subsequence of the sequence $\{\tau_n\}$. The times between consecutive registrations, $\theta_n' = \tau_{n+1}' - \tau_n'$, are also identically distributed, independent, positive random variables with distribution function $R(x)$. In this section we shall compute the mean of $R(x)$.

In this particle counting framework, the random variable $\bar{\xi}_n = (\xi_n^1, \dots , \xi_n^s)$ denotes the number of impulses of each type present in the counter just before the n-th particle arrives. In particular, $P_{0\dots0}^{(n)}$ is the probability that the n-th arrival finds the counter free and is registered. The limit

$$\lim_{n \to \infty} P_{0\dots0}^{(n)} = P_{0\dots0}$$

exists and is given by

39) $$P_{0\dots0} = \sum_{r_1=0}^{\infty} \dots \sum_{r_s=0}^{\infty} (-1)^{\sum_{i=1}^{s} r_i} B_{r_1\dots r_s} ,$$

where the $B_{r_1\dots r_s}$ are given by (36). By applying Wald's Fundamental Identity of Sequential Analysis (see [2] page 183), we conclude that the mean time between successive registrations is

40) $$\frac{\alpha}{P_{0\dots0}} .$$

Before we know what kind of particle has arrived, the duration of the impulse produced by the particle has the distribution

$$H(x) = 1 - \sum_{i=1}^{s} p_i \, e^{-\mu_i x} \quad . \tag{41}$$

Therefore, the time between consecutive registrations in a Type II counter, when the particles arrive according to a recurrent process and the impulse times are distributed as a weighted sum of exponentially distributed random variables, and the time between consecutive registrations in our process ~~has~~ have the same distribution. Hence

$$\frac{\alpha}{P_{0\cdots0}}$$

is also the mean time between successive registrations in this second process. Since any distribution function can be approximated by one of the form (41), this result may be of practical value. Unfortunately, we have not been able to obtain the variance of $R(x)$ by the use of the imbedded Markov chain.

References

[1] Takács, L. (1958) On a Coincidence Problem Concerning Telephone Traffic 9 Acta Math. 45-81 .

[2] Takács, L. (1962) Introduction to the Theory of Queues. Oxford Univ. Press, New York.